A Decidable Spatial Logic With Cone-shaped Cardinal Directions

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CSL'09

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What is this talk about?

Shortly

This talk is about **deciding satisfiability** of formulas from a suitable **modal logic** under interpretation over **labeled rational planes** $\mathcal{L} : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathscr{P}(A)$.

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What is this talk about?

An example – Compass Logic (Venema '90)

Formulas of **Compass Logic** are defined by the following grammar:

$$
\begin{array}{rccccccc}
\varphi & := & a & | & \neg\varphi & | & \varphi \vee \varphi & | & \varphi \wedge \varphi & | \\
& & \diamondsuit\!\!\diagdown\,\varphi & | & \diamondsuit\!\!\diagup\,\varphi & | & \diamondsuit\!\!\diagdown\,\varphi & | & \diamondsuit\!\!\diagup\,\varphi & | \\
& & \boxminus\varphi & | & \boxed{!}\,\varphi & | & \boxminus\varphi & | & \boxed{\cdot}\,\varphi.
\end{array}
$$

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$$\boxminus \varphi \quad | \quad \boxed{\uparrow} \varphi \quad | \quad \boxminus \varphi \quad | \quad \boxed{\downarrow} \varphi .$$

Formulas are evaluated over *labeled points* of the rational planes...

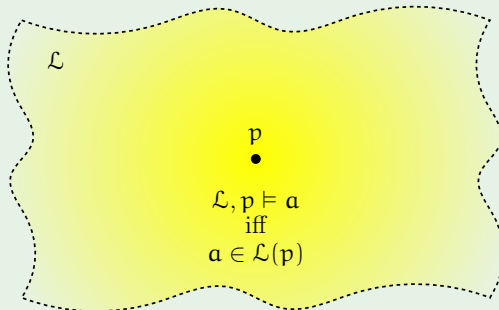| Introduction	Cone Logic	Satisiability	Interval logics	Outline
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\end{aligned}
$$

Formulas are evaluated over *labeled points* of the rational planes...

$$\mathcal{L}$$

$$p$$
$$\bullet$$

$$\mathcal{L}, p \vDash a$$
iff
$$a \in \mathcal{L}(p)$$

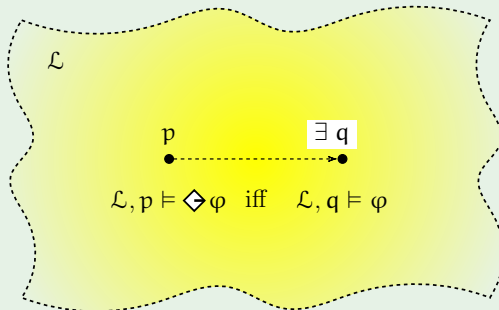| Introduction	Cone Logic	Satisiability	Interval logics	Outline
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& & \boxminus\varphi \quad | \quad \boxdot\varphi \quad | \quad \boxminus\varphi \quad | \quad \boxdot\varphi.
\end{array}
$$

Formulas are evaluated over *labeled points* of the rational planes...

$$\mathcal{L}, p \vDash \diamondsuit\varphi \quad \text{iff} \quad \mathcal{L}, q \vDash \varphi$$

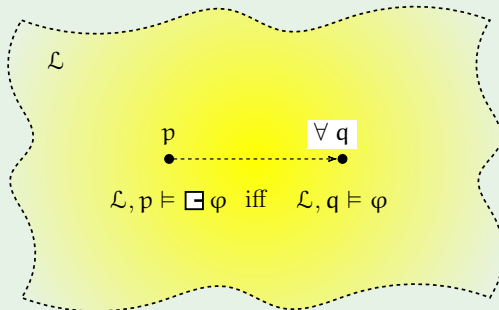| Introduction	Cone Logic	Satisiability	Interval logics	Outline
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Formulas are evaluated over *labeled points* of the rational planes...

$$\mathcal{L}, p \vDash \boxminus \varphi \quad \text{iff} \quad \mathcal{L}, q \vDash \varphi$$

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What is this talk about?

Satisfiability problem

The **satisfiability problem** consists of deciding, given a formula φ, whether there exist *a labeled structure* $\mathcal{L}$ and *a point* p such that

$$\mathcal{L}, p \vDash \varphi.$$

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What is this talk about?

Satisfiability problem

The **satisfiability problem** consists of deciding, given a formula φ, whether there exist *a labeled structure* $\mathcal{L}$ and *a point* p such that

$$\mathcal{L}, p \models \varphi.$$

Unfortunately...

Theorem (Marx and Reynolds '99)

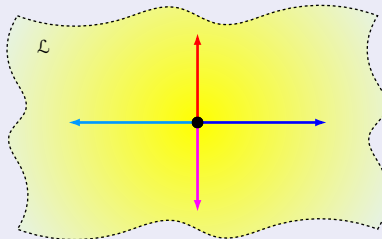The satisfiability problem for Compass Logic is **undecidable**.

(one can encode an infinite tiling using $\diamondsuit$, $\diamondsuit$, $\diamondsuit$, $\diamondsuit$...)

Decidability may be recovered by **weakening** Compass Logic...

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Weakening Compass Logic

Cone Logic as a weakening of Compass logic

Instead of having modalities for the *positive/negative* x-/y-axes...
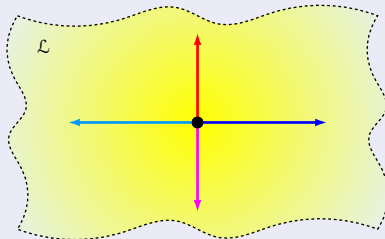

existential
modalities:

universal
modalities:

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Weakening Compass Logic

Cone Logic as a weakening of Compass logic

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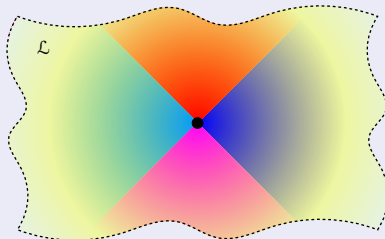
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...we introduce modalities for *cone-shaped regions*:

Cone Logic as a weakening of Compass logic

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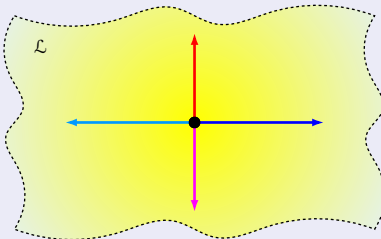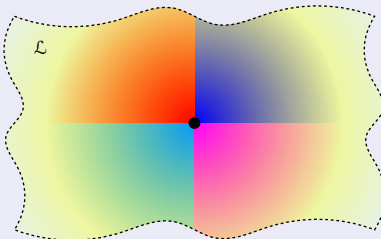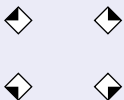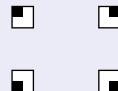
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...we introduce modalities for *cone-shaped regions*:

existential
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Syntax

Cone Logic formulas

Formulas of **Cone Logic** are defined by the following grammar:

$$\varphi \ := \ a \quad | \quad \neg\varphi \quad | \quad \varphi \vee \varphi \quad | \quad \varphi \wedge \varphi \quad |$$
$$\Diamondleftblack \varphi \quad | \quad \Diamondright \varphi \quad | \quad \Diamonddown \varphi \quad | \quad \Diamondrightblack \varphi \quad |$$
$$\Boxleftblack \varphi \quad | \quad \Boxright \varphi \quad | \quad \Boxdown \varphi \quad | \quad \Boxrightblack \varphi$$

The modal operators $\Diamondleftblack$, $\Diamondright$, $\Diamonddown$, $\Diamondrightblack$, $\Boxleftblack$, $\Boxright$, $\Boxdown$, $\Boxrightblack$ quantify over points of the four **open** quadrants, e.g.,

$$\mathcal{D}om(\Diamondleftblack) \ = \ \big\{ q = (x, y) : x > 0, \, y > 0 \big\}.$$

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Syntax

Cone Logic formulas

Formulas of **Cone Logic** are defined by the following grammar:

$$\varphi \; := \; a \; | \; \neg\varphi \; | \; \varphi \vee \varphi \; | \; \varphi \wedge \varphi \; |$$
$$\diamondsuit\!\!\!\blacktriangleleft\,\varphi \; | \; \diamondsuit\!\!\!\blacktriangleright\,\varphi \; | \; \diamondsuit\!\!\!\blacktriangledown\,\varphi \; | \; \diamondsuit\,\varphi \; |$$
$$\blacksquare\!\square\,\varphi \; | \; \square\!\blacksquare\,\varphi \; | \; \square\!\!\blacksquare\,\varphi \; | \; \square\!\!\blacksquare\,\varphi \; |$$
$$\diamondsuit\!\!\!\blacktriangleleft^{+}\,\varphi \; | \; \diamondsuit\!\!\!\blacktriangleright^{+}\,\varphi \; | \; \diamondsuit\!\!\!\blacktriangledown^{+}\,\varphi \; | \; \diamondsuit^{+}\,\varphi \; |$$
$$\blacksquare\!\square^{+}\,\varphi \; | \; \square\!\blacksquare^{+}\,\varphi \; | \; \square\!\!\blacksquare^{+}\,\varphi \; | \; \square\!\!\blacksquare^{+}\,\varphi$$

The modal operators $\diamondsuit\!\!\blacktriangleleft$, $\diamondsuit\!\!\blacktriangleright$, $\diamondsuit\!\!\blacktriangledown$, $\diamondsuit$, $\blacksquare\!\square$, $\square\!\blacksquare$, $\square\!\!\blacksquare$, $\square\!\!\blacksquare$
quantify over points of the four **open** quadrants, e.g.,

$$\mathcal{D}om\big(\diamondsuit\!\!\blacktriangleleft\big) \; = \; \big\{ q = (x, y) : x > 0, \, y > 0 \big\}.$$

The modal operators $\diamondsuit\!\!\blacktriangleleft^{+}$, $\diamondsuit\!\!\blacktriangleright^{+}$, $\diamondsuit\!\!\blacktriangledown^{+}$, $\diamondsuit^{+}$, $\blacksquare\!\square^{+}$, $\square\!\blacksquare^{+}$, $\square\!\!\blacksquare^{+}$, $\square\!\!\blacksquare^{+}$
quantify over points of the four **semi-closed** quadrants, e.g.,

$$\mathcal{D}om\big(\diamondsuit\!\!\blacktriangleleft^{+}\big) \; = \; \big\{ q = (x, y) : x \geqslant 0, \, y \geqslant 0 \big\} \setminus \big\{ (0, 0) \big\}.$$

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Expressiveness

Cone Logic makes it easy to express spatial relationships
based on (approximate) cardinal directions...

Example 1

"For every pair of points p *and* q *labeled, respectively,
by* a *and* b, q *is to the North-East of* p*."*
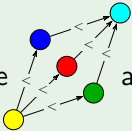
is expressed by the Cone Logic formula

$$\varphi \ = \ \blacksquare(a \ \rightarrow \ \blacksquare\neg b \ \wedge \ \square\neg b \ \wedge \ \square\neg b)$$

where $\blacksquare$ is a shorthand for $\blacksquare \square$
(equivalent to *"for every point of the plane"*).

...and it can also express also interesting properties of the plane!

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Expressiveness

Example 2

Let A be the lattice and consider the Hintikka-like formula

$$\varphi_A = \blacksquare \bigvee_{a \in A} a \ \wedge \ \blacksquare \bigwedge_{a \neq b} \neg (a \wedge b)$$
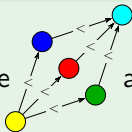
$$\wedge \ \blacksquare \bigwedge_{a \in A} \left(a \rightarrow \bigwedge_{b \leqslant a} \blacklozenge b \ \wedge \ \bigwedge_{b \geqslant a} \diamondsuit b \ \wedge \ \square \bigvee_{b \leqslant a} b \ \wedge \ \square \bigvee_{b \geqslant a} b \right)$$

$\mathcal{L}$

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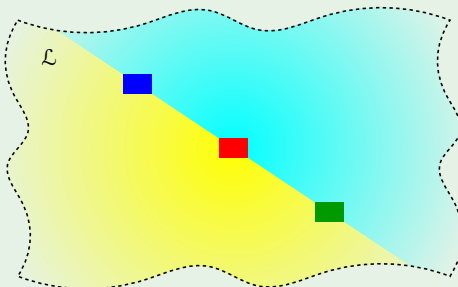○○○○

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Expressiveness

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$$\wedge \ \blacksquare \bigwedge_{a \in A} \left(a \ \rightarrow \ \bigwedge_{b \leqslant a} \blacklozenge b \ \wedge \ \bigwedge_{b \geqslant a} \Diamond b \ \wedge \ \square \bigvee_{b \leqslant a} b \ \wedge \ \square \bigvee_{b \geqslant a} b \right)$$

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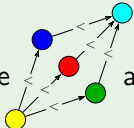Expressiveness

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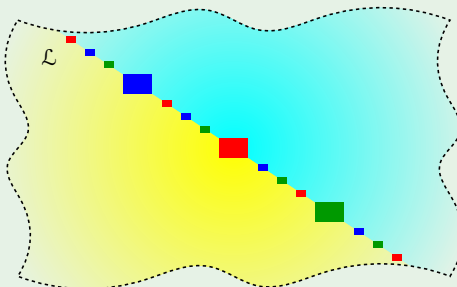| Introduction	Cone Logic	Satisiability	Interval logics	Outline
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To solve the satisfiability problem for Cone Logic,
we consider *portions* of the rational plane:

Stripe

A **stripe** of a labeled rational plane $\mathcal{L} : \mathbb{Q} \times \mathbb{Q} \rightarrow A$ is the restriction $\mathcal{L}_{[x_0, x_1]}$ of $\mathcal{L}$ to a region of the form $[x_0, x_1] \times \mathbb{Q}$.
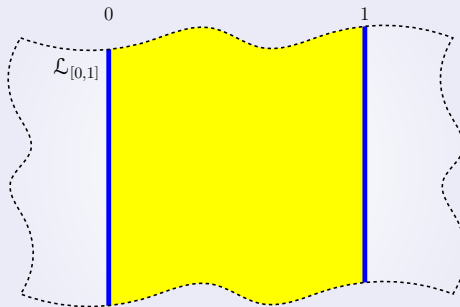
Fact

Any Cone Logic formula φ can be translated into a formula $\varphi_{[x_0, x_1]}$ in such a way that, for every labeled rational plane $\mathcal{L}$,

$$\exists\, p \in \mathbb{Q} \times \mathbb{Q}. \qquad\qquad \exists\, p \in [x_0, x_1] \times \mathbb{Q}.$$
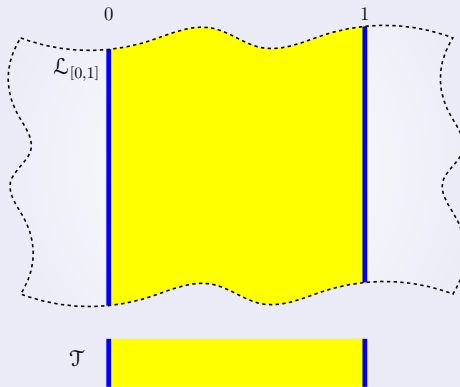$$\mathcal{L}, p \vDash \varphi \qquad \text{iff} \qquad \mathcal{L}_{[x_0, x_1]}, p \vDash \varphi_{[x_0, x_1]}.$$

$\Rightarrow$ We can restrict our attention to satisfiability over a stripe $\mathcal{L}_{[0,1]}$ (and we forget, for the moment, the operators $\langle\!\!\blacktriangleleft^{+}\!\!\rangle$, $\langle\!\!\blacklozenge^{+}\!\!\rangle$, ...).

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Decompositions
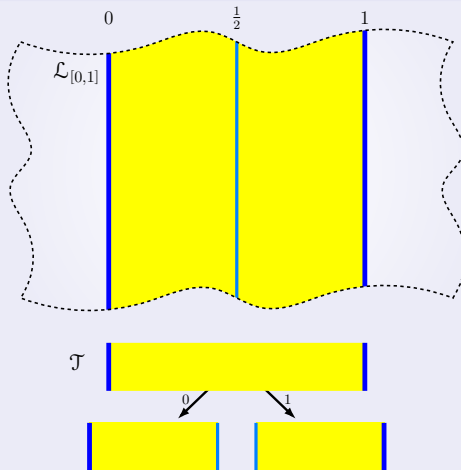
Decompositions of stripes

By exploiting isomorphism between the orders
over $[0, 1]$ and over $\left\{ \frac{i}{2^n} : n \in \mathbb{N}, 0 \leqslant i \leqslant 2^n \right\}$,
we decompose the stripe $\mathcal{L}_{[0,1]}$ into a **tree structure** $\mathcal{T}$...

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Decompositions of stripes

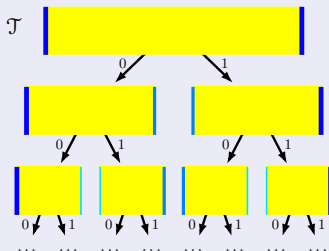| Introduction	Cone Logic	**Satisiability**	Interval logics	Outline
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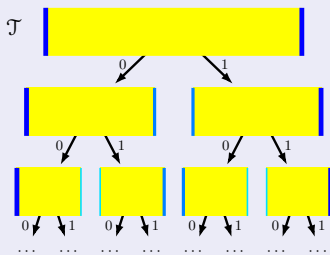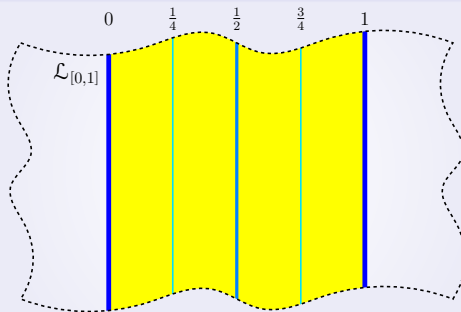
Decompositions

Decompositions of stripes

...In such a way, we can get rid of the interiors of the (sub-)stripes and focus on the formulas (of a certain bounded complexity) that hold along their borders.

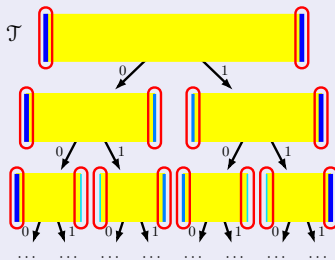## Decompositions of stripes

...In such a way, we can get rid of the interiors of the (sub-)stripes and focus on the formulas (of a certain bounded complexity) that hold along their borders.

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Filtration

Finite abstractions of stripes

Given a (sub-)stripe $\mathcal{L}_{[x_0, x_1]}$, we define an equivalence $\approx$ over $\mathbb{Q}$ such that $y \approx y'$ iff, for all formulas α (of bounded complexity),

$$\mathcal{L}, (x_0, y) \vDash \begin{array}{l} \diamondsuit\!\!\!\!\blacktriangleleft \alpha \ / \ \blacksquare\!\!\!\!\lrcorner \alpha \ / \\ \diamondsuit\!\!\!\!\triangleright \alpha \ / \ \square\!\!\!\!\llcorner \alpha \end{array} \qquad \mathcal{L}, (x_1, y) \vDash \begin{array}{l} \diamondsuit\!\!\!\!\triangleright \alpha \ / \ \blacksquare\!\!\!\!\llcorner \alpha \ / \\ \diamondsuit\!\!\!\!\triangleleft \alpha \ / \ \square\!\!\!\!\lrcorner \alpha \end{array}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\mathcal{L}, (x_0, y') \vDash \begin{array}{l} \diamondsuit\!\!\!\!\blacktriangleleft \alpha \ / \ \blacksquare\!\!\!\!\lrcorner \alpha \ / \\ \diamondsuit\!\!\!\!\triangleright \alpha \ / \ \square\!\!\!\!\llcorner \alpha \end{array} \qquad \mathcal{L}, (x_1, y') \vDash \begin{array}{l} \diamondsuit\!\!\!\!\triangleright \alpha \ / \ \blacksquare\!\!\!\!\llcorner \alpha \ / \\ \diamondsuit\!\!\!\!\triangleleft \alpha \ / \ \square\!\!\!\!\lrcorner \alpha \end{array}$$

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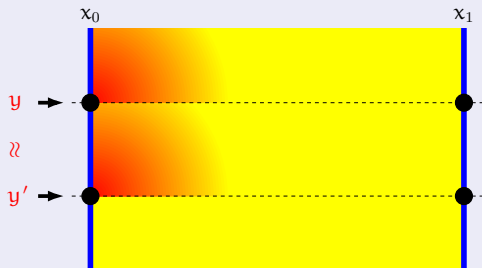
$$\mathcal{L},(x_0,y) \vDash \begin{array}{c} \Diamond\!\!\blacktriangleright\alpha \,/\, \blacksquare\!\!\!\square\,\alpha \,/ \\ \Diamond\alpha \,/\, \square\!\!\blacksquare\,\alpha \end{array} \qquad \mathcal{L},(x_1,y) \vDash \begin{array}{c} \Diamond\alpha \,/\, \blacksquare\!\!\square\,\alpha \,/ \\ \Diamond\!\!\blacktriangleright\alpha \,/\, \square\!\!\blacksquare\,\alpha \end{array}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\mathcal{L},(x_0,y') \vDash \begin{array}{c} \Diamond\!\!\blacktriangleright\alpha \,/\, \blacksquare\!\!\square\,\alpha \,/ \\ \Diamond\alpha \,/\, \square\!\!\blacksquare\,\alpha \end{array} \qquad \mathcal{L},(x_1,y') \vDash \begin{array}{c} \Diamond\alpha \,/\, \blacksquare\!\!\square\,\alpha \,/ \\ \Diamond\!\!\blacktriangleright\alpha \,/\, \square\!\!\blacksquare\,\alpha \end{array}$$

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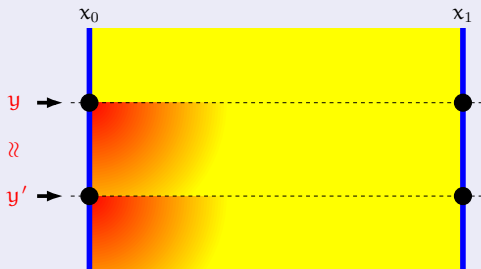
$$\mathcal{L}, (x_0, y) \models \begin{array}{l} \diamondsuit\!\!\!\blacktriangleleft\, \alpha \,/\, \boxminus\,\alpha \,/ \\ \textcolor{red}{\diamondsuit}\,\alpha \,/\, \textcolor{red}{\boxminus}\,\alpha \end{array} \qquad \mathcal{L}, (x_1, y) \models \begin{array}{l} \diamondsuit\,\alpha \,/\, \boxminus\,\alpha \,/ \\ \diamondsuit\,\alpha \,/\, \boxminus\,\alpha \end{array}$$

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Filtration

Finite abstractions of stripes

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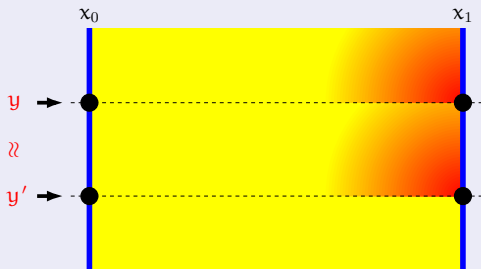
$$\mathcal{L}, (x_0, y) \models \begin{array}{l} \diamondsuit\!\!\!\blacktriangleleft\, \alpha \ / \ \blacksquare\!\!\!\square\, \alpha \ / \\ \diamondsuit\!\!\!\triangleright\, \alpha \ / \ \square\!\!\!\square\, \alpha \end{array}$$

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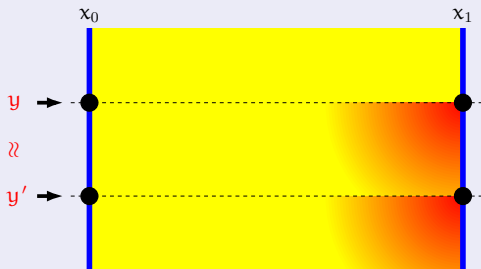
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Tree pseudo-model property

Since the equivalence relation $\approx$ has *finite index*, we have that

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Proposition (a tree pseudo-model property)

Any given stripe $\mathcal{L}_{[0,1]}$ can be represented by means of a suitable *infinite binary tree* $\mathcal{T}$ whose vertices are labeled over a finite alphabet (we call the structure $\mathcal{T}$ a **tree decomposition** of $\mathcal{L}_{[0,1]}$).

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(we call the structure $\mathcal{T}$ a **tree decomposition** of $\mathcal{L}_{[0,1]}$).

...However, tree decompositions must be properly constrained
so that they correctly represents some concrete stripes.

Examples of constraints on a tree decomposition

- For every pair of *sibling* vertices $v = [x_0, x_1]$ and $v' = [x'_0, x'_1]$ in $\mathcal{T}$, the labeling of the right border of v has to *match* with the labeling of the left border of v'
  (in such a way, we can assume $x_1 = x'_0$),

- There is *no infinite path* π in $\mathcal{T}$ such that, for every vertex $v \in \pi$, $\Diamond\!\!\!\!\lozenge\, \alpha$ appears on the *left border* of v and neither $\Diamond\!\!\!\!\lozenge\, \alpha$ nor α appear on the *right border* of v.

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Reduction to a CTL fragment

Theorem 1 (reduction to a CTL fragment)

Constrained tree decompositions can be defined
in a **fragment of CTL**, which we denote **CTL⁻**.

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Reduction to a CTL fragment

Theorem 1 (reduction to a CTL fragment)

Constrained tree decompositions can be defined
in a **fragment of CTL**, which we denote **CTL$^-$**.

Theorem 2 (deciding satisfiability of CTL$^-$)

The satisfiability problem for CTL$^-$ ($\Rightarrow$ Cone Logic) is in **PSPACE**.

Proof idea

CTL$^-$ formulas are *conjunctions* of the following basic formulas:

1. $\mathbf{AG}(left \vee right)$, $\mathbf{AG}\neg(left \wedge right)$, $\mathbf{AG}(\mathbf{EX}left \wedge \mathbf{EX}right)$

2. $\mathbf{AG}(\alpha)$, $\mathbf{AG}(\alpha \rightarrow \mathbf{AF}\alpha')$,
   where α, α' contains only positive occurrences of
   the modal operator $\mathbf{AX}$ and no other operators.

$\Rightarrow$ Checking satisfiability of these formulas amounts
   at deciding **universality of Büchi automata**.

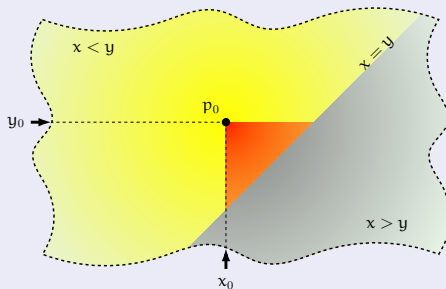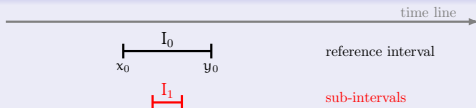| Introduction	Cone Logic	Satisiability	Interval logics	Outline
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From points to intervals

There is a natural correspondence between intervals $I = [x, y]$, with $x < y$, and points $p = (x, y)$ of the rational plane...

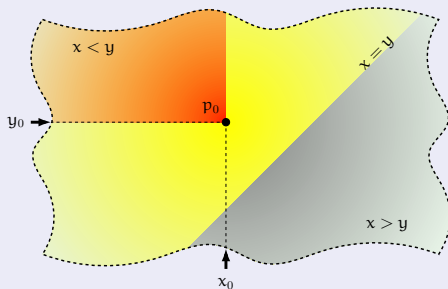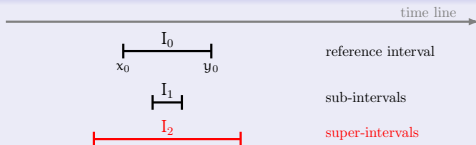### Spatial interpretation of intervals and their relationships

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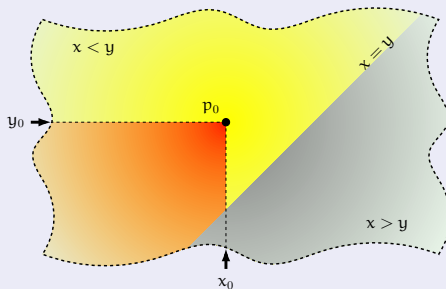Spatial interpretation of intervals and their relationships

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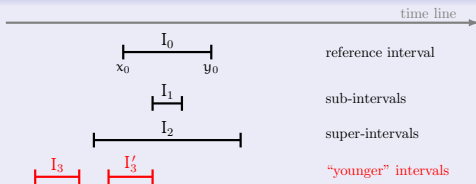### Spatial interpretation of intervals and their relationships

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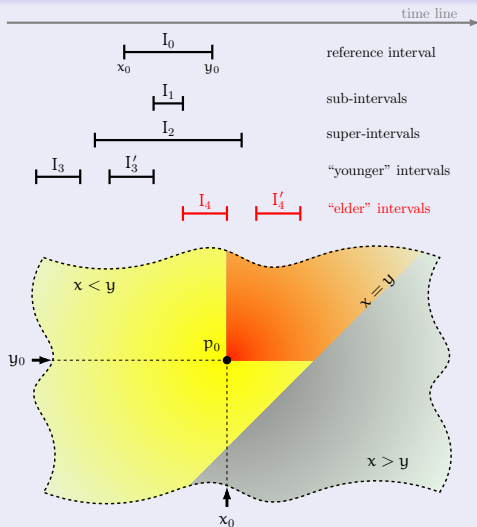### Spatial interpretation of intervals and their relationships

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Spatial interpretation of intervals and their relationships

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An interesting interval logic

Cone Interval Logic

We can introduce a new **interval temporal logic**
by means of the following grammar:

$$\varphi := \quad a \quad | \quad \neg\varphi \quad | \quad \varphi \vee \varphi \quad | \quad \varphi \wedge \varphi \quad |$$
$$\langle sub \rangle \varphi \quad | \quad \langle super \rangle \varphi \quad | \quad \langle younger \rangle \varphi \quad | \quad \langle elder \rangle \varphi \quad |$$
$$[sub]\varphi \quad | \quad [super]\varphi \quad | \quad [younger]\varphi \quad | \quad [elder]\varphi.$$

Formulas are evaluated over *labeled intervals* of the rational line.

Note: *sub* and *super* correspond to Allen's interval relations
D and $\bar{D}$, while *younger* and *elder* are unions of other Allen's
interval relations (e.g., *elder* $= O \cup E \cup A \cup L$).

Moreover, Cone Interval Logic generalizes previous interval
temporal logics (cf. [Bresolin, Goranko, Montanari, Sala '08]).

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Definability in Cone Logic

Fact 1

The modal operators $\langle sub \rangle$, $\langle super \rangle$, $\langle younger \rangle$, $\langle elder \rangle$, ... are definable by means of the modal operators ◇, ◆, ◆, ◆, ...
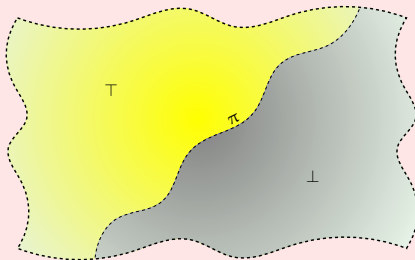
Fact 1

The modal operators $\langle sub \rangle$, $\langle super \rangle$, $\langle younger \rangle$, $\langle elder \rangle$, ... are
definable by means of the modal operators $\Diamond$, $\Diamond$, $\Diamond$, $\Diamond$, ...

Fact 2

The "yellow region" of valid intervals (i.e., $\{p = (x, y) : x < y\}$)
is definable (up to "deformations") inside the rational plane:

$$\blacksquare(\top \vee \bot \vee \pi) \wedge \blacksquare(\neg\top \vee \neg\bot) \wedge \blacksquare(\neg\top \vee \neg\pi) \wedge \blacksquare(\neg\bot \vee \neg\pi) \wedge$$
$$\blacksquare(\top \rightarrow \Diamond\top \wedge \blacksquare\top) \wedge \blacksquare(\bot \rightarrow \Diamond\bot \wedge \blacksquare\bot) \wedge$$
$$\blacksquare(\pi \rightarrow \blacksquare^{+}\top \wedge \blacksquare^{+}\bot) \wedge \blacksquare(\Diamond\pi \wedge \Diamond\pi \rightarrow \pi \vee \Diamond\pi \vee \Diamond\pi).$$

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Satisfiability

From previous results we have:

Corollary (decidability of Cone Interval Logic)

The satisfiability problem for Cone Interval Logic is in **PSPACE**.

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Satisfiability

From previous results we have:

Corollary (decidability of Cone Interval Logic)

The satisfiability problem for Cone Interval Logic is in **PSPACE**.

Theorem (Shapirovsky and Shehtman '03)

The satisfiability problem for the fragment of Cone Interval Logic that comprised only the modal operators $\langle sub \rangle$ and $[sub]$ is **PSPACE-hard**.

$\Rightarrow$ This also implies that the decision procedure for satisfiability of Cone Logic formulas is **optimal**.

Summary

In conclusion, Cone Logic

- is a **weakening** of Venema's Compass Logic,
- has a **PSPACE-complete** satisfiability problem,
- subsumes interesting **interval temporal logics**.

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Possible generalizations?

- spaces with **more than** 2 **dimensions** (e.g., $\mathbb{Q}^3$),
- modal operators based on **"narrow" cones**,
- satisfiability with different underlying orders
  (e.g., **real plane**, **discrete grid**).

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Thank you!